# Geometric Ruzsa triangle inequality in metric spaces with dilations

Marius Buliga

Institute of Mathematics, Romanian Academy
P.O. BOX 1-764, RO 014700
Bucureşti, Romania
Marius.Buliga@imar.ro

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**Abstract**

This is a short note related to the article arXiv:1212.5056 [math.CO] "On growth in an abstract plane" by Nick Gill, H. A. Helfgott, Misha Rudnev. It contains a general "geometric Ruzsa triangle inequality" which is then applied to get the same kind of inequality in a metric space with dilations.

**Motivation.** In arXiv:1212.5056[math.CO] "On growth in an abstract plane" by Nick Gill, H. A. Helfgott, Misha Rudnev, in lemma 4.1 is given a proof of the Ruzsa triangle inequality which intrigued me. Later on, at the end of the article the authors give a geometric Ruzsa inequality in a Desarguesian projective plane, based on similar ideas as the ones used in the proof of the Ruzsa triangle inequality. All this made me write the following.

**Results.** Let $X$ be a non-empty set and $\Delta : X \times X \to X$ be an operation on $X$ which has the following two properties:

1. for any $a, b, c \in X$ we have $\Delta(\Delta(a, b), \Delta(a, c)) = \Delta(b, c)$,
2. for any $a \in X$ the function $z \mapsto \Delta(z, a)$ is injective.

We may use weaker hypotheses for $\Delta$, namely:

1. (weaker) there is a function $F : X \times X \to X$ such that $F(\Delta(a, b), \Delta(a, c)) = \Delta(b, c)$ for any $a, b, c \in X$,
2. (weaker) there is a function $G : X \times X \to X$ such that $a \mapsto G(\Delta(a, b), b)$ is an injective function for any $b \in X$.

**Proposition 1** *Let $X$ be a non empty set endowed with an operation $\Delta$ which satisfies 1. and 2. (or the weaker version of those). Then for any non empty sets $A, B, C \subset X$ there is an injection*

$$i : \Delta(C, A) \times B \to \Delta(B, C) \times \Delta(B, A)$$

*where we denote by $\Delta(A, B) = \{\Delta(a, b) \mid a \in A, b \in B\}$*
*In particular, if $A, B, C$ are finite sets, we have the Rusza triangle inequality*

$$| \Delta(C, A) || B | \leq | \Delta(B, C) || \Delta(B, A) |$$

*where $| A |$ denotes the cardinality of the finite set $A$.*

I shall give the proof for hypotheses 1., 2., because the proof is the same for the weaker hypotheses. Also, this is basically the same proof as the one of the mentioned lemma 4.1. The proof of the Ruzsa inequality corresponds to the choice $\Delta(a,b) = -a+b$, where $(X,+)$ is a group (no need to be abelian). The proof of the geometric Ruzsa inequality corresponds to the choice $\Delta(a,b) = [b,a]$, with the notations from the article, with the observation that this function $\Delta$ satisfies weaker 1. and 2.

**Proof.** We can choose functions $f : \Delta(C,A) \to C$ and $g : \Delta(C,A) \to A$ such that for any $x \in \Delta(C,A)$ we have $x = \Delta(f(x), g(x))$. With the help of these functions let

$$i(x,b) = (\Delta(b, f(x)), \Delta(b, g(x))) \qquad .$$

We want to prove that $i$ is injective. Let $(c,d) = i(x,b) = i(x',b')$. Then, by 1. we have $x = x' = \Delta(c,d)$. This gives an unique $e = f(x) = f(x')$. Now we know that $\Delta(b,e) = \Delta(b, f(x)) = c = \Delta(b', f(x')) = \Delta(b', e)$. By 2. we get that $b = b'$ qed.

I shall use this proposition in the frame of metric spaces with dilations $(X,d,\delta)$, where:

- there is no algebraic structure, like the one of a group, except the approximate operations provided by the field of dilations,
- nor any incidence structure, like in a projective space, for example.

Concerning metric spaces with dilations I introduced these spaces under the name "dilatation structures" in the article arXiv:math/0608536 [math.MG] "Dilatation structures I. Fundamentals". In particular regular sub-riemannian manifolds, riemannian manifolds and Lie groups with a left invariant distance induced by a completely non-integrable (i.e. generating) distribution are examples of such spaces. For the most advanced presentation see the course notes arXiv:1206.3093[math.MG] "Sub-riemannian geometry from intrinsic viewpoint".

In a metric space with dilations $(X,d,\delta)$ we have the function approximate difference $\Delta^e_\varepsilon(a,b)$ based at $e \in X$ and applied to a pair of closed points $a,b \in X$. This function has the property that $(e,a,b) \mapsto \Delta^e_\varepsilon(a,b)$ converges uniformly (on compact sets) to $\Delta^e(a,b)$ as $\varepsilon$ goes to 0. Moreover, there is a local group operation with $e$ as neutral element such that $\Delta^e(a,b) = -a+b$, therefore the function $\Delta^e$ satisfies 1. and 2. (The local group operation is "conical", in particular for the case of sub-riemannian manifolds is a Carnot group operation.)

The approximate difference operation $\Delta^e_\varepsilon$ satisfies the following approximate version of property 1.:

1. (approximate) for any $e,a,b,c \in X$ which are sufficiently close and for any $\varepsilon \in (0,1)$ we have, with the notation $a(\varepsilon) = \delta^e_\varepsilon a$, the relation

$$\Delta^{a(\varepsilon)}_\varepsilon(\Delta^e_\varepsilon(a,b), \Delta^e_\varepsilon(a,c)) = \Delta^e_\varepsilon(b,c) \qquad .$$

We say that a set $A \subset X$ is $\varepsilon$ separated if for any $x,y \in A$, the inequality $d(x,y) < \varepsilon$ implies $x = y$. Further I am going to write about sets which are closed to a fixed, but arbitrary otherwise point $e \in X$.

**Proposition 2** *In a metric space with dilations, let $p > 0$ and let $A, B, C$ be finite sets of points included in a compact neighbourhood of $e$, which are closed to $e \in X$, such that for any $\varepsilon \in (0,p)$ the sets $B$ and $\Delta^e_\varepsilon(C,A)$ are $\mu$ separated. Then for any $\varepsilon \leq C(\mu)$ there is an injective function*

$$i : \Delta^e_\varepsilon(C,A) \times B \to \Delta^e_\varepsilon(B,C) \times \Delta^e_\varepsilon(B,A) \qquad .$$

**Proof.** As previously, we choose the functions $f$ and $g$. Notice that these functions depend on $\varepsilon$ but this will not matter further. I shall use the $O(\varepsilon)$ notation liberally, for example $y = x + O(\varepsilon)$ means $d(x, y) \leq O(\varepsilon)$. Let's define the function $i$ by the same formula as previously:

$$i(x, b) = (\Delta^e_\varepsilon(b, f(x)), \Delta^e_\varepsilon(b, g(x))) \qquad .$$

Let $(x, b)$ and $(x', b')$ be pairs such that $i(x, b) = i(x', b') = (c_\varepsilon, d_\varepsilon)$. From 1. (approximate) and from the uniform convergence mentioned previously we get that

$$x = x' + O(\varepsilon) = \Delta^e(c_\varepsilon, d_\varepsilon) + O(\varepsilon) \qquad .$$

There is a function $C(\mu)$ such that $\varepsilon \leq C(\mu)$ implies (the last from the previous relation) $O(\varepsilon) < \mu$. For such a $\varepsilon$, by the separation of $\Delta^e_\varepsilon(C, A)$ we get $x = x'$.

Let $z = f(x)$. From the hypothesis we have $\Delta^e_\varepsilon(b, z) = \Delta^e_\varepsilon(b', z)$. This implies, via the structure of the function $\Delta^e$ and via the uniform convergence, that $b' = b + O(\varepsilon)$ (by compactness, this last $O(\varepsilon)$ does not depend on $A, B, C$). By the same reasoning as previously, we may choose $C(\mu)$ such that $d(b, b') < \mu$ if $\varepsilon \leq C(\mu)$. This implies $b = b'$ qed.

**Remark.** The contents of this note appeared first in the blog post Geometric Ruzsa triangle inequalities and metric spaces with dilations.

# References


[1] M. Buliga, Dilatation structures I. Fundamentals, *J. Gen. Lie Theory Appl.*, **1** (2007), 2, 65-95.

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